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On Convergence Classes in L-fuzzy Topological Spaces*

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Convergence theory is a primary topic in topology. In fact, topology and so-called convergence class are characterized by each other. In fuzzy topology (L-fuzzy topology), more than 40 papers published in the last ten years were concerned with convergence theory. Among these papers, the problem of convergence class was solved for the case of $L = [0, 1]$ [7]. Since the neighbor structure, so called "quasi-coincident neighborhood system," of an L-fuzzy point in an L-fuzzy topological space is in general not directed under the inclusion order, the conditions of convergence class in $[0, 1]$-fuzzy topology will not be valid any longer in the case of lattice. Moreover, quite different from the cases of $\{0, 1\}$-fuzzy topology (i.e., ordinary topology) and $[0, 1]$-fuzzy topology, the so called Bolzano–Weierstrass property does not hold, i.e., a net with a cluster point in an L-fuzzy topological space is not still necessary to have a subnet converging to the point. In this paper, a necessary and sufficient condition for the Bolzano–Weierstrass property is produced, the result is also used in a satisfactory theory of convergence classes in L-fuzzy topological spaces, and the associated characterization theorem between L-fuzzy topologies and convergence classes is established. © 1998 Academic Press

Key Words: net; convergence; convergence class; L-fuzzy topological space

1. PRELIMINARIES

In the sequel, X always stands for a nonempty ordinary set, L always means a completely distributive lattice with an ordering-reversing involution $'$: $L \to L$, i.e., a fuzzy lattice (F-lattice for short). The smallest element and the largest element of L are denoted by 0_L and 1_L, or 0 and 1 for short, respectively. For every $A \in L^X$, denote $\text{supp}(A) = \{x \in X: A(x) > 0\}$, call it the *support* of A. Also define $A' \in L^X$ by $A'(x) = A(x)'$ for

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every $x \in X$. For every $a \in L$, let $\underline{a}$ denote the constant mapping from X to L with value a.

Note that the partial order $\leq$ on L naturally induces a pointwise order $\leq$ on L^X as follows: For every pair $U, V \in L^X$,

$$u \leq V \Leftrightarrow \forall x \in X, \qquad U(x) \leq V(x).$$

This order is also a partial order, and it makes L^X to be a F-lattice. So for convenience, we always consider every subfamily $\mathscr{A} \subset L^X$ has been equipped with the relative order in L^X.

DEFINITION 1.1. A relation $\leq$ on a set P is called a *preorder*, if $\leq$ is reflexive and transitive. A set P equipped with a preorder $\leq$ is called a *preordered set*.

A relation $\leq$ on a set D is called *directed*, if for every finite $D_0 \subset D$, there exists $d_0 \in D$ such that $d \leq d_0$ for every $d \in D_0$. A set D equipped with a directed preorder $\leq$ is called a *directed set* or *up-directed set*. The dual notion *down-directed set* is dually defined.

DEFINITION 1.2. $\forall a \in L$, denote

$$\uparrow a = \{b \in L \colon b \geq a\}, \downarrow a = \{b \in L \colon b \leq a\}.$$

$\alpha \in L$ is called *join-irreducible*, if for every two $a, b \in L$,

$$\alpha \leq a \vee b \Rightarrow \alpha \leq a \text{ or } \alpha \leq b;$$

every nonzero join-irreducible element is called a *molecule*. For every $A \subset L$, denote the set of all the molecules contained in A by $M(A)$.

Note that for a nonempty ordinary set X, a F-lattice L and an L-fuzzy subset $A \in L^X$, according to the symbol defined above, $\downarrow A$ is the set of all the elements in L^X smaller than A. So the set of all the molecules of L^X smaller than A is just

$$M(\downarrow A) = \{x_\lambda \colon x \in X, \lambda \in M(L), \lambda \leq A(x)\}.$$

DEFINITION 1.3. $\forall x \in X$, $a \in L$, denote the L-fuzzy subset taking value a at x and value 0 at other points of X by x_a, call it an *L-fuzzy point* on X. For every $\mathscr{A} \subset L^X$, denote the set of all the L-fuzzy points on X contained in $\mathscr{A}$ by $Pt(\mathscr{A})$; especially, for every $A \in L^X$, $Pt(\downarrow A)$ means the set of all the L-fuzzy points contained in $\downarrow A$, i.e., smaller than A.

$\delta \subset L^X$ is called an *L-fuzzy topology* on X, if δ is closed under arbitrary joins and finite meets; especially, $\mathbf{0}, \mathbf{1} \in \delta$. Call (L^X, δ) an *L-fuzzy topological space*, or call it an L-fts for short. Every $U \in \delta$ is called an *open subset* in (L^X, δ), and every $P \in L^X$ such that $P' \in \delta$ is called a *closed subset* in (L^X, δ). Denote the family of all the closed subsets in (L^X, δ) by δ'.

DEFINITION 1.4. Let (L^X, δ) be an L-fts, $A \in L^X$. Define respectively the *interior* A^0 and the *closure* A^- of A as

$$A^0 = \bigvee \{U \in \delta : U \leq A\}, \qquad A^- = \bigwedge \{P \in \delta' : P \geq A\}.$$

For every two $A, B \in L^X$, say A *quasi-coincides with* B, denoted by $A\hat{q}B$, if there exists $x \in X$ such that $A(x) \nleq B'(x)$; otherwise, denote it by $A \neg \hat{q}B$.

Let $x_a \in Pt(L^X)$, $U \in \delta$, $P \in \delta'$. U is called a *quasi-coincident neighborhood* of x_a, if $x_a \hat{q} U$; denote the family of all the quasi-coincident neighborhoods of x_a by $\mathscr{Q}(x_a)$, called the *quasi-coincident neighborhood system* of x_a in (L^X, δ). $x_\lambda \in M(L^X)$ is called an *adherent point* of $A \in L^X$, if $U\hat{q}A$ for every $U \in \mathscr{Q}(x_\lambda)$.

According to the previous stipulation, every quasi-coincide neighborhood system $\mathscr{Q}(x_a)$ in (L^X, δ) is equipped with the relative order in L^X.

DEFINITION 1.5. Define a relation $\preccurlyeq$ on L as follows: For every two $a, b \in L$, $b \preccurlyeq a$ if and only if for every $C \subset L$ such that $\bigvee C \geq a$, there exists $c \in C$ such that $b \leq c$. Denote $\beta(a) = \{b \in L : b \preccurlyeq a\}$, Every subset $D \subset \beta(a)$ satisfying $\bigvee D = a$ is called a *minimal set* of a in L.

THEOREM 1.6 [6, 11–13]. *Let L be a complete lattice. Then the following conditions are equivalent*:

(i) *L is completely distributive.*

(ii) *Every element of L has a minimal set.*

(iii) *Every element of L has a minimal set consisting of molecules in L.*

COROLLARY 1.7. *Every element in a completely distributive lattice can represented as a join of molecules.*

THEOREM 1.8. *Let* (L^X, δ) *be an L-fts. Then for every* $x_\lambda \in M(L^X)$, $\mathscr{Q}(x_\lambda)$ *is a down-directed set in* L^X *and* $\underline{0} \notin \mathscr{Q}(x_a)$.

PROPOSITION 1.9. *Let* (L^X, δ) *be an L-fts,* $A, B, C \in L^X$, $\{A_t : t \in T\} \subset L^X$, $x \in X$, $a \in L \backslash \{0\}$. *Then*

(i) $A\hat{q}B \Leftrightarrow B\hat{q}A \Leftrightarrow A \nleq B' \Leftrightarrow B \nleq A'$.

(ii) $A\hat{q} \bigvee_{t \in T} \Leftrightarrow \exists t \in T,\ A\hat{q}A_t$.

(iii) $A \leq B,\ C\hat{q}A \Rightarrow C\hat{q}B$.

DEFINITION 1.10. Let X be a nonempty ordinary set, L a F-lattice, a mapping $c: L^X \to L^X$ is called a *closure operator* on L^X, if it fulfills the following conditions:

(CO1) $c(0) = \underline{0}$.

(CO2) $\forall A \in L^X,\ A \leq c(A)$.

(CO3) $\forall A, B \in L^X,\ c(A \vee B) = c(A) \vee c(B)$.

(CO4) $\forall A \in L^X,\ c(c(A)) = c(A)$.

For a closure operator c on L^X, define the L-fuzzy topology *generated by* c as

$$\delta = \{A \in L^X : c(A') = A'\}.$$

THEOREM 1.11. *Let X be a nonempty ordinary set, L a F-lattice, c: $L^X \to L^X$ an operator on L^X satisfying conditions $(CO1)$–$(CO3)$, then the L-fuzzy topology δ generated by c is exactly an L-fuzzy topology on X. Moreover, if c also fulfills condition $(CO4)$, then $A^- = c(A)$ holds in the L-fts (L^X, δ) for every $A \in L^X$.*

THEOREM 1.12. *Let X be a nonempty ordinary set, L a F-lattice, $\mathscr{C}$ the family of all the closure operators on L^X, $\mathscr{T}$ the family of all the L-fuzzy topologies on X. Then*

$$f: \mathscr{C} \to \mathscr{T}, \qquad f(c) = \{A \in L^X : c(A') = A'\}$$

is a bijection, and its reverse is just

$$f^{-1}: \mathscr{T} \to \mathscr{I}, \qquad f^{-1}(\delta) = cl_\delta.$$

2. NET AND CONVERGENCE

DEFINITION 2.1. Let D be a directed set, $A \in L^X$. Call every mapping S: $D \to Pt(\downarrow A)$ a *net* in A, and D the *index set* of S. Especially, a mapping S: $D \to M(\downarrow A)$ a *molecule net* in A. Also call a net (a molecule net, respectively) in $\underline{1}$ *a net in L^X* (*a molecule net in L^X*, respectively).

A net S in L^X with index set D is also denoted by S: $D \to (L^X, \delta)$ or $S = \{S(n), n \in D\}$.

For a net $S = \{S(n), n \in D\}$ in L^X and $e \in Pt(L^X)$, S is called a *constant net* with value e, if $S(n) = e$ for every $e \in D$.

DEFINITION 2.2. Let (L^X, δ) be an L-fts, $S = \{S(n), n \in D\}$ a net in (L^X, δ), P a property, $e \in Pt(L^X)$.

Call S *eventually* possesses the property P, if there exists $n_0 \in D$ such that for every $n \in D$, $n \geq n_0$, $S(n)$ always possess the property P. Call S *frequently* possesses the property P, if for every $n \in D$, there always exist $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)$ possesses the property P.

Call e a *cluster point* of S, denoted by $S \infty e$, if for every Q-neighborhood U of e, S frequently quasi-coincides with U. Call e a *limit point* or a *limit* for short, denoted by $S \to e$, if for every Q-neighborhood U of e, S eventually quasi-coincides with U; in this case we also say S *converges* to e, or say S is *convergent* to e.

Denote the join of all the cluster points of S by clu S, the join of all the limit points of net S by lim S.

THEOREM 2.3. *Let (L^X, δ) be an L-fts, $S = \{S(n),\ n \in D\}$ a net in (L^X, δ), $e \in Pt(L^X)$. Then*

(i) $S \to e \Rightarrow S\infty e$.

(ii) $\lim S \leq \operatorname{clu} S$.

(iii) $S\infty e \geq d \Rightarrow S\infty d$.

(iv) $S \to e \geq d \Rightarrow S \to d$.

(v) $S\infty e \Leftrightarrow e \leq \operatorname{clu} S$.

(vi) $S \to e \Leftrightarrow e \leq \lim S$.

DEFINITION 2.4. Let (L^X, δ) be an L-fts, S: $D \to (L^X, \delta)$, T: $E \to (L^X, \delta)$ be two nets in (L^X, δ). Call T is a *subnet* of S, if there exists a mapping N: $E \to D$, called a *cofinal selection* on S, such that

(i) $T = S \circ N$;

(ii) For every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$ for $m \geq m_0$.

THEOREM 2.5. *Let (L^X, δ), (L^Y, μ) be an L-fts, S a net in (L^X, δ), T a subset of S, $e \in Pt(L^X)$. Then*

(i) $S \to e \Rightarrow T \to e$.

(ii) $\lim S \leq \lim T$.

(iii) $T\infty e \Rightarrow S\infty e$.

(iv) $\operatorname{clu} T \leq \operatorname{clu} S$.

THEOREM 2.6. *Let (L^X, δ) be an L-fts, S a net in (L^X, δ), $e \in Pt(L^X)$. Then $S \to e$ if and only if $T\infty e$ for every subnet T of S.*

Proof. (Necessity) By Theorems 2.5(i) and 2.3(i).

(Sufficiency) Suppose $S = \{S(n),\ n \in D\}$, S does not converge to e. Then $\exists U_0 \in \mathscr{Q}(e)$ such that S frequently does not quasi-coincide with U_0. That is to say, $\exists U_0 \in \mathscr{Q}(e)$, $\forall n \in D$, $\exists N(n) \in D$ such that $N(n) \geq n$, $S(N(n)) \neg\, \hat{q} U_0$. So we get a cofinal selection N: $D \to D$ and then a subnet $T = S \circ N$ of S. Since for $U_0 \in \mathscr{Q}(e)$ and every $n \in D$, $T(n) = S(N(n)) \neg\, \hat{q} U_0$, e is not a cluster point of T.

About a cluster point of a net, we have furthermore the following important theorem.

THEOREM 2.7. *Let (L^X, δ) be an L-fts, $e \in Pt(L^X)$. Then the following conditions are equivalent*:

(i) *$\mathscr{Q}(e)$ is down-directed.*

(ii) *Bolzano–Weierstrass property*: *For every net S in (L^X, δ) such that $S \infty e$, S has a subnet $T \to e$.*

(iii) *For every molecule net S in (L^X, δ) such that $S \infty e$, S has a subnet $T \to e$.*

Proof. (i) ⇒ (ii). Suppose $\mathscr{Q}(e)$ is down-directed and net $S \infty e$ in (L^X, δ), hope to prove that S has a subnet $T \to e$.

Suppose D is the index set of S, $\mathscr{A}$ is a down-directed down-cofinal subset of $\mathscr{Q}(e)$. Take $E = D \times \mathscr{A}$, define a relation $\leq$ on E by

$$\forall (m, U), (n, V) \in E, \qquad (m, U) \leq (n, V) \Leftrightarrow m \leq n, \qquad u \geq V,$$

Then $\leq$ is a directed preorder (in fact, a directed partial order) on E, E equipped with $\leq$ is a directed set.

$\forall (n, U) \in E$, since $U \in \mathscr{Q}(e)$, $S \infty e$, $\exists N(n, U) \in D$ such that $N(n, U) \geq n$, $S(N(n, U)) \hat{q} U$. So we get a mapping N: $E \to D$. $\forall n_0 \in D$, since $\mathscr{A}$ is a down-directed set, $\mathscr{A} \neq \emptyset$, $\exists U_0 \in \mathscr{A}$. So $(n_0, U_0) \in E$. Then $\forall (n, U) \in E$ such that $(n, U) \geq (n_0, U_0)$, we have $N(n, U) \geq n \geq n_0$. That is to say N: $E \to D$ is a cofinal selection, $T = S \circ N$ is a subnet of S.

$\forall U \in \mathscr{Q}(e)$, since $\mathscr{A}$ is down-directed in $\mathscr{Q}(e)$, $\exists U_0 \in \mathscr{A}$ such that $U_0 \leq U$. Since $S \infty e$, $\exists n_0 \in D$ such that $S(n_0) \hat{q} U_0$. Then $(n_0, U_0) \in E$. $\forall (n, V) \in E$ such that $(n, V) \geq (n_0, U_0)$, we have $T(n, V) = S(N(n, V)) \hat{q} V \leq U_0, T(n, V) \hat{q} U_0$. So T eventually quasi-coincides with $U_0, T \to e$.

(ii) ⇒ (iii): Obvious.

(iii) ⇒ (i): We prove the implication in three steps:

(1) Construct index set D. Denote the set of all natural numbers by $\mathbf{N}$, let

$$D = \{(U_1, \dots, U_r) : r \in \mathbf{N}, \forall i \in \{1, \dots, r\}, U_i \in \mathscr{Q}(e)\},$$

define a relation $<$ on D as follows: For arbitrary $(U_1, \dots, U_r), (V_1, \dots, V_s) \in D$,

$$(U_1, \dots, U_r) < (V_1, \dots, V_s) \Leftrightarrow \{U_1, \dots, U_r\} \subsetneqq \{V_1, \dots, V_s\}.$$

Finally, define a relation $\leq$ on D by

$$\forall m, n \in D, \qquad m \leq n \Leftrightarrow m < n \text{ or } m = n.$$

Then one can easily find that $\leq$ is a preorder on D; in fact, $\leq$ is even a partial order on D. Now for arbitrary $(U_1, \ldots, U_r), (V_1, \ldots, V_s) \in D$, we have the following relation in D:

$$(U_1, \ldots, U_r), (V_1, \ldots, V_s) \leq (U_1, \ldots, U_r, V_1, \ldots, V_s) \in D.$$

So that D equipped with $\leq$ is a directed set.

(2) Construct net $S = \{S(n),\ n \in D\}$. $\forall n = (U_1, \ldots, U_r) \in D$, then $U_1 \in \mathscr{Q}(e)$, $e\hat{q}U_1$. Since $e \in Pt(L^X)$, we can assume $e = x_a$. By Corollary 1.7, there exists $C \subset M(L)$ such that $\bigvee C = a$. Then by $x_a\hat{q}U_1$ we have $\bigvee C = a \not\leq U_1(x)'$, there exists $\lambda \in C$ such that $\lambda \not\leq U_1(x)'$, $x_\lambda\hat{q}U_1$. Therefore, take $S(n) = x_\lambda$, we have a molecule net $S = \{S(n), n \in D\}$ in (L^X, δ) satisfying $S((U_1, \ldots, U_r)\hat{q}U_1$ for every $(U_1, \ldots, U_r) \in D$.

(3) Prove $\mathscr{Q}(e)$ is down-directed. First of all, we prove $S\infty e$. In fact, let $U \in \mathscr{Q}(e)$ and $n = (U_1, \ldots, U_r) \in D$. Take $m = (U, U_1, \ldots, U_r) \in D$, then according to the preorder $\leq$ defined in (1), $m \geq n$; according to the definition of S in (2), $S(m)\hat{q}U$. So $S\infty e$. Now following the assumption in (iii), S has a subnet $T = \{T(m),\ m \in E\} \to e$. Let $U, V \in \mathscr{Q}(e)$, going to prove that $U \wedge V \in \mathscr{Q}(e)$. Since $T \to e$, $\exists k \in E$ such that $T(m)$ quasi-coincides with both U and V for every $m \geq k$. Since T is a subnet of the molecule net S, $T(m)$ is also a molecule in (L^X, δ), and hence $T(m)$ quasi-coincides with $U \wedge V$. But according to the definition of every $S(n)$ in (2), as one of elements in $\{S(n): n \in D\}$, it follows $T(m) \leq e$, and hence $e\hat{q}(U \wedge V), U \wedge V \in \mathscr{Q}(e)$. This completes the proof. ∎

Remark 2.8. The implication (iii) ⇒ (i) in Theorem 2.7 has another proof as follows, which is longer but maybe more geometrical:

Let A be a nonempty set, ρ an ordinal number. By Zermelo's Theorem on well-ordering, there exists a relation $<_w$ well-ordering A. Define relation $\leq_w$ on A by

$$\forall a, b \in A, \qquad a \leq_w b \Leftrightarrow a <_w b \text{ or } a = b$$

and relation $\leq$ on $\rho \times A$ by

$$\forall (\alpha, a), (\beta, b) \in \rho \times A, \qquad (\alpha, a) \leq (\beta, b) \Leftrightarrow \alpha < \beta, \text{or } \alpha = \beta, \\ a \leq_w b.$$

Call the set $\rho \times A$ equipped with this relation $\leq$ the *well-ordered ρ-copy* of A. Then for every nonempty set A and every ordinal number ρ, the relation $\leq$ on the well-ordered ρ-copy $\rho \times A$ well-orders $\rho \times A$, i.e., $\rho \times A$ equipped with $\leq$ is a well-ordered set. Suppose $\mathscr{Q}(e)$ is not down-directed, then $\exists U_0, U_1 \in \mathscr{Q}(e)$ such that $U_0 \wedge U_1 \notin \mathscr{Q}(e)$. Take D as the well-ordered ω-copy $\omega \times \mathscr{Q}(e)$ of $\mathscr{Q}(e)$, then D is a well-ordered set and then clearly a directed set.

Suppose $e = x_a$. $\forall (n, U) \in D$, there totally exist the following five possibilities:

Case (1). $U_0 \wedge U \in \mathscr{Q}(x_a)$, $U_1 \wedge U \notin \mathscr{Q}(x_a)$;
Case (2). $U_0 \wedge U \notin \mathscr{Q}(x_a)$, $U_1 \wedge U \in \mathscr{Q}(x_a)$;
Case (3). $U_0 \wedge U \notin \mathscr{Q}(x_a)$, $U_1 \wedge U \notin \mathscr{Q}(x_a)$;
Case (4). $U_0 \wedge U \in \mathscr{Q}(x_a)$, $U_1 \wedge U \in \mathscr{Q}(x_a)$, $\exists m < \omega$, $n = 2m$;
Case (5). $U_0 \wedge U \in \mathscr{Q}(x_a)$, $U_1 \wedge U \in \mathscr{Q}(x_a)$, $\exists m < \omega$, $n = 2m + 1$.

We shall take $S(n, U) \in M(L^X)$ such that

$$\left.\begin{array}{llll} \text{In Case (1)} & S(n,U)\hat{q}U, & S(n,U)\neg\,\hat{q}U_1; & \\ \text{In Case (2)} & S(n,U)\hat{q}U, & S(n,U)\neg\,\hat{q}U_0; & \\ \text{In Case (3)} & S(n,U)\hat{q}U, & S(n,U)\neg\,\hat{q}U_0, & S(n,U)\neg\,\hat{q}U_1; \\ \text{In Case (4)} & S(n,U)\hat{q}(U_0 \wedge U), & S(n,U)\neg\,\hat{q}U_1; & \\ \text{In Case (5)} & S(n,U)\hat{q}(U_1 \wedge U), & S(n,U)\neg\,\hat{q}U_0. & \end{array}\right\} \tag{2.1}$$

For Case (1), since $U \in \mathscr{Q}(x_a)$, $a \nleq U(x)'$. Since $a = \bigvee M(\downarrow a)$, so $\exists \lambda \in M(\downarrow a)$ such that $\lambda \nleq U(x)'$. Since $U_1 \wedge U \notin \mathscr{Q}(x_a)$, $\lambda \leq a \leq (U_1 \wedge U)'(x) = U_1(x)' \vee U(x)'$. By $\lambda \in M(L)$ and $\lambda \nleq U(x)'$, we have $\lambda \leq U_1(x)'$. So we take $S(n, U) = x_\lambda \in M(L^X)$, then $S(n,U)\hat{q}U$, $S(n,U)\neg\,\hat{q}U_1$.

For Case (2), similar to Case (1), we can take $S(n, U) \in M(L^X)$ such that $S(n,U)\hat{q}U, S(n,U)\neg\,\hat{q}U_0$.

For Case (3), similar to Case (1), we can take $S(n, U) \in M(L^X)$ such that $S(n,U)\hat{q}U, S(n,U)\neg\,\hat{q}U_0, U_1$.

For Case (4), since $U_0 \wedge U \in \mathscr{Q}(x_a)$, $a \nleq (U_0 \wedge U)'(x)$. Since $a = \bigvee M(\downarrow a)$, $\exists \lambda \in M(\downarrow a)$ such that $\lambda \nleq (U_0 \wedge U)'(x)$. By the supposition at the beginning of this proof of (vii) $\Rightarrow$ (i), $U_0 \wedge U_1 \notin \mathscr{Q}(x_a)$, so

$$\begin{aligned} \lambda \leq a &\leq (U_0 \wedge U_1)'(x) = (U_0' \vee U_1')(x) \leq \big((U_0 \wedge U)' \vee U_1'\big)(x) \\ &= (U_0 \wedge U)'(x) \vee U_1'(x). \end{aligned}$$

By $\lambda \in M(L)$ and $\lambda \nleq (U_0 \wedge U)'(x)$, we have $\lambda \leq U_1'(x)$. Take $S(n, U) = x_\lambda$, then $S(n,U)\hat{q}(U_0 \wedge U)$ but $S(n,U)\neg\,\hat{q}U_1$.

For Case (5), similar to Case (4), we can take $S(n, U) \in M(L^X)$ such that $S(n,U)\hat{q}(U_1 \wedge U)$ but $S(n,U)\neg\,\hat{q}U_0$.

Now S: $D \to M(L^X)$ is a molecule net in (L^X, δ) satisfying relations in group (2.1).

$\forall U \in \mathscr{Q}(e)$, $\forall (n, V) \in D$, then $(2(n+1), U) \in D$, $(2(n+1), U) > (n, V)$ and by relations (2.1), $S(2(n+1), U)\hat{q}U$ [in Case (1) to (3)], or

$S(2(n+1), U)\hat{q}(U_0 \wedge U)$ [in Case (4)] and then $S(2(n+1), U)\hat{q}U$, or $S(2(n+1), U)\hat{q}(U_1 \wedge U)$ [in Case (5)] and then $S(2(n+1), U)\hat{q}U$. So $S\infty e$.

If $T: E \to M(L^X)$ is a subnet of S such that $T \to e$. Then for $U_0, U_1 \in \mathscr{Q}(e)$, there should exist $m_0, m_1 \in E$ such that $\forall m \in E$, $m \geq m_0 \Rightarrow T(m)\hat{q}U_0$, $m \geq m_1 \Rightarrow T(m)\hat{q}U_1$. Since E is directed, $\exists m_2 \in E$ such that $m_2 \geq m_0, m_1$. So $T(m)\hat{q}U_0, U_1$ for every $m \in E, m \geq m_2$. But T takes values in img(S), $\exists(n, U) \in D$ such that $T(m) = S(n, U)$. By relations (2.1), there is no $S(n, U)$ can quasi-coincide with both U_0 and U_1 simultaneously. So T cannot be convergent to e, S has not subnet which converges to e.

COROLLARY 2.1. *Let (L^X, δ) be an L-fts, S a net in (L^X, δ), $e \in M(L^X)$. Then $S\infty e$ if and only if S has a subnet $T \to e$.*

3. NET, CLOSEDNESS AND CONTINUITY

THEOREM 3.1. *Let (L^X, δ) be an L-fts, $e, A \in L^X$, If $e \in Pt(L^X)$, then the following conditions are equivalent*:

(i) $e \leq A^-$.

(ii) *There exists a net S in A such that $S\infty e$.*

(iii) *There exists a molecule net S in A such that $S\infty e$.*

If $e \in M(L^X)$, then the following conditions are equivalent to condition (i) *mentioned above*:

(iv) *e is an adherent point of A.*

(v) *There exists a net S in A such that $S \to e$.*

(vi) *There exists a molecule net S in A such that $S \to e$.*

Proof. Suppose $e = x_a \in Pt(L^X)$.

(i) $\Rightarrow$ (iii). Take D as the well-ordered ω-copy $\omega \times \mathscr{Q}(e)$ of $\mathscr{Q}(e)$. $\forall(n,U) \in D$, then $U \in \mathscr{Q}(x_a)$, $\bigvee M(\downarrow a) = a \nleq U(x)'$, $\exists \lambda \in M(\downarrow a)$, $\lambda \nleq U(x)'$, $U \in \mathscr{Q}(x_\lambda)$. Since $x_\lambda \leq x_a \leq A^-$, x_λ is an adherent point of A, for $U \in \mathscr{Q}(x_\lambda)$, $A \nleq U'$. So $\exists y_\gamma \in M(\downarrow A)$ such that $y_\gamma \nleq U'$. Denote $S(n, U) = y_\gamma$, we have $S(n, U) \leq A$, $S(n, U)\hat{q}U$. So we have obtained a molecule net $S = \{S(n, U), (n, U) \in D\}$ in A. $\forall U \in \mathscr{Q}(e)$, $\forall(n, V) \in D$, $(n+1, U) \in D$, $(n+1, U) \geq (n, V)$, $S(n+1, U)\hat{q}U$. So S frequently quasi-coincides with U. Hence $S\infty e$.

(iii) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (i). If there exists a net $S = \{S(n), n \in D\}$ in A such that $S\infty x_a$, then $\forall \lambda \in M(\downarrow a)$, $\forall U \in \mathscr{Q}(x_\lambda)$, $U \in \mathscr{Q}(x_a)$. $\forall n \in D$, by $S\infty x_a$, $\exists n_0 \in D$ such that $n_0 \geq n$, $S(n_0)\hat{q}U$, $S(n_0) \nleq U'$. Since S is in A, $S(n_0) \leq A$, so

$A \not\leq U'$, $U\hat{q}A$. By the arbitrariness of $U \in \mathscr{Q}(x_\lambda)$, x_λ is an adherent point of A. So $x_\lambda \leq A^-$, $\lambda \leq A^-(x)$. By the arbitrariness of $\lambda \in M(\downarrow a)$, $a = \bigvee M(\downarrow a) \leq A^-(x)$, $x_a \leq A^-$.

Now suppose $e \in M(L^X)$.

(vi) $\Rightarrow$ (v). Obvious.

(v) $\Rightarrow$ (iv). By Theorem 2.3(i) and what are proved above, (v) $\Rightarrow$ (ii) $\Rightarrow$ (i). Then since $e \in M(L^X)$, (i) $\Rightarrow$ (iv) is obtained.

(iv) $\Rightarrow$ (i). By the property of adherent points.

(i) $\Rightarrow$ (vi). We have had (i) $\Rightarrow$ (iii). On the other hand, since $e \in M(L^X)$, by Corollary 2.1, (iii) $\Rightarrow$ (iv).

COROLLARY 3.2. *Let* (L^X, δ) *be an L-fts,* $A \in L^X$. *Then the following conditions are equivalent*:

(i) $A = A^-$.

(ii) *For every net S in A and every* $e \in Pt(L^X)$, $S \infty e \Rightarrow e \leq A$.

(iii) *For every molecule net S in A and every* $e \in Pt(L^X)$, $S \infty e \Rightarrow e \leq A$.

(iv) *For every net S in A and every* $e \in M(L^X)$, $S \to e \Rightarrow e \leq A$.

(v) *For every molecule net S in A and every* $e \in M(L^X)$, $S \to e \Rightarrow e \leq A$. ∎

4. CONVERGENCE CLASSES

In Ref. [9], Liu and Pu introduced the notion of fuzzy convergence classes, and proved that the fuzzy topologies and fuzzy convergence classes were completely determined by each other. Furthermore, Liu pointed out [7] that the characterization of fuzzy convergence classes given in Ref. [9] was not perfect, and a correct condition was given at the same time. This new condition is just for reflecting the stratification structure of the value domain $L = [0, 1]$ there. In L-fuzzy topology, the new condition is then invalid again; so finding a proper condition to replace it is just the aim of this section.

DEFINITION 4.1. Let L^X be an L-fuzzy space, D a directed set, $\{E^n: n \in D\}$ a family of directed sets, $S^n = \{S^n(m), m \in E^n\}$ a net in $Pt(L^X)$ for every $n \in D$. Then for the product directed set $\mathbf{D} = D \times \Pi_{n \in D} E^n$, the net $\mathbf{S}: \mathbf{D} \to L^X$ defined as

$$\forall (n, f) \in \mathbf{D}, \qquad \mathbf{S}(n, f) = S^n(f(n))$$

is called the *induced* net of the net family $\{S^n: n \in D\}$.

DEFINITION 4.2. Let $\mathscr{A} \subset L^X$.

Let $\mathscr{S}_M(\mathscr{A})$ denote the class of all the nets $S = \{S(n), n \in D\}$ such that $S(n) \in M(\mathscr{A})$ for every $n \in D$. Let $\mathscr{C} \subset \mathscr{S}_M(L^X) \times M(L^X)$. Let $(S, e) \in \mathscr{S}_M(L^X) \times M(L^X)$. Say S *$\mathscr{C}$-converges to e*, denoted by $S \to_{\mathscr{C}} e$, if $(S, e) \in \mathscr{C}$. Denote the case that S does not $\mathscr{C}$-converge to e by $S \nrightarrow_{\mathscr{C}} e$.

$\mathscr{C}$ is called an *L-fuzzy convergence class* on L^X, if it fulfills the following conditions:

(i) If $S \in \mathscr{S}_M(L^X)$ is a constant net with value $e \in M(L^X)$, then $S \to_{\mathscr{C}} e$. (CC1)

(ii) If $S \to_{\mathscr{C}} e$ and T is a subnet of S, then $T \to_{\mathscr{C}} e$. (CC2)

(iii) For every $(S, e) \in \mathscr{S}_M(L^X) \times M(L^X)$, if $S \nrightarrow_{\mathscr{C}} e$, then there exists a subnet T of S such that for every subnet R of T, $R \nrightarrow_{\mathscr{C}} e$. (CC3)

(iv) For every directed set D and every $\{S^n\colon n \in D\} \subset \mathscr{S}_M(L^X)$, where $S^n = \{S^n(m), m \in E^n\}$ and $S^n \to_{\mathscr{C}} S(n)$ for every $n \in D$, if $S \to_{\mathscr{C}} e$ for the obtained molecule net $S = \{S(n), n \in D\}$, then for the induced net $\vec{S}$ of $\{S^n\colon n \in D\}$, $\vec{S} \to_{\mathscr{C}} e$. (CC4)

(v) For every $x \in X$, every $A \subset M(L)$ and every molecule $\lambda \le \bigvee A$, there exists $S \in \mathscr{S}_M(\{x_\xi\colon \xi \in A\})$ such that $S \to_{\mathscr{C}} x_\lambda$. (CC5)

DEFINITION 4.3. Let (L^X, δ) be an L-fts. Denote

$$\varphi(\delta) = \left\{(S, e) \in \mathscr{S}_M(L^X) \times M(L^X)\colon S \to e\right\}.$$

Call $\varphi(\delta)$ the *L-fuzzy convergence class on L^X generated by δ.*

THEOREM 4.4. *Let (L^X, δ) be an L-fts, then $\varphi(\delta)$ is an L-fuzzy convergence class on L^X.*

Proof. Verifications are straightforward. We verify (CC5) for instance.

Suppose $x \in X$, $\mathscr{A} \subset M(L)$, $\lambda \in M(L)$, $\lambda \le \bigvee A$. $\forall U \in \mathscr{Q}(x_\lambda)$, then $\lambda \not\le U(x)'$. Since $\lambda \le \bigvee A$, we have $\bigvee A \not\le U(x)'$, $\exists \lambda_U \in A$ such that $\lambda_U \not\le U(x)'$, $\lambda_U \hat{q} U$. Take $S(U) = x_{\lambda_U}$. Since $x_\lambda \in M(L^X)$, $D = \mathscr{Q}(e)$ equipped with the partial order $\le_D$ defined as

$$\forall U, V \in D, \qquad U \le_D V \Leftrightarrow U \ge V$$

is a directed set. So we obtain a molecule net $S = \{S(U), U \in D\} \in \mathscr{S}_M(\{x_\xi\colon \xi \in A\})$ and clearly $S \to e$. ∎

LEMMA 4.5. *Let $\mathscr{C} \subset \mathscr{S}_M(L^X) \times M(L^X)$ fulfill* (CC4) *and* (CC5), $\{(S^t, e^t)\colon t \in T\} \subset C$, $e \in M(L^X)$, $e \le \bigvee_{t \in T} e^t$. *Then there exists* $\vec{S} \in \mathscr{S}_M(\bigcup_{t \in T} \operatorname{img}(S^t))$ *such that* $\vec{S} \to_{\mathscr{C}} e$.

Proof. Suppose $e = x_\lambda$, $A = \{\xi \in M(L)\colon \exists t \in T,\ x_\xi = e^t\}$, then we have $\lambda \leq \bigvee A$ by $e \leq \bigvee_{t \in T} e^t$. By (CC5), there exists $S = \{S(n), n \in D\} \in \mathscr{S}_M(\{x_\xi\colon \xi = A\})$ such that $S \to_{\mathscr{C}} x_\lambda$. Certainly $S(n) \in \{x_\xi\colon \xi \in A\} \subset \{e^t\colon t \in T\}$ for every $n \in D$, so $\forall n \in D$, $\exists t(n) \in T$ such that $S^{t(n)} \to_{\mathscr{C}} e^{t(n)} = S(n)$. Then by (CC4), the induced net $\vec{S}$ of $\{S^{t(n)}\colon n \in D\}$ $\mathscr{C}$-converges to e. Clearly we have $\vec{S} \in \mathscr{S}_M(\bigcup_{t \in T} \operatorname{img}(S^t))$. ∎

LEMMA 4.6. *Let $\mathscr{C} \subset \mathscr{S}_M(L^X) \times M(L^X)$ fulfill* (CC4) *and* (CC5), *$\mathscr{A} \subset M(L^X)$, $S \in \mathscr{S}_M(\downarrow \bigvee \mathscr{A})$, $S \to_{\mathscr{C}} e$. Then there exists $\mathbf{S} \in \mathscr{S}_M(\mathscr{A})$ such that $\mathbf{S} \to_{\mathscr{C}} e$.*

Proof. Suppose the index set of S is D. $\forall n \in D$, since $S(n) \in M(\downarrow \bigvee \mathscr{A})$, by (CC5), similar to the proof of Lemma 4.7 one can easily prove that $\exists T^n \in \mathscr{S}_M(\mathscr{A})$ such that $T^n \to_{\mathscr{C}} S(n)$. Denote the induced net of $\{T^n\colon n \in D\}$ by $\mathbf{S}$, then $\mathbf{S} \in \mathscr{S}_M(\mathscr{A})$ and by (CC4), $\mathbf{S} \to_{\mathscr{C}} e$.

DEFINITION 4.7. Let $\mathscr{C} \subset \mathscr{S}_M(L^X) \times M(L^X)$.
For every $A \in L^X$, denote

$$\operatorname{clu}_{\mathscr{C}}(A) = \left\{e \in M(L^X)\colon \exists S \in \mathscr{S}_M(\downarrow A),\ S \to_{\mathscr{C}} e\right\}.$$

Define an operator c on L^X, called the *closure operator generated by $\mathscr{C}$*, as follows:

$$c(A) = \bigvee \operatorname{clu}_{\mathscr{C}}(A), \qquad \forall A \in L^X.$$

THEOREM 4.8. *Let X be a nonempty ordinary set, L a F-lattice, $\mathscr{C}$ an L-fuzzy convergence class on L^X. Then the closure operator c generated by $\mathscr{C}$ is a closure operator on L^X.*

Proof. We are going to verify the conditions (CO1)–(CO4) of a closure operator.

(CO1). Clear.

(CO2). By (CC1).

(CO3). Let $A, B \in L^X$. By the definition of c, clearly $A \leq B \Rightarrow c(A) \leq c(B)$, so we need only prove $\operatorname{clu}_{\mathscr{C}}(A \vee B) \subset \operatorname{clu}_{\mathscr{C}}(A) \cup \operatorname{clu}_{\mathscr{C}}(B)$.

$\forall e \in \operatorname{clu}_{\mathscr{C}}(A \vee B)$, then $\exists S \in \mathscr{S}_M(A \vee B)$ such that $S \to_{\mathscr{C}} e$. Suppose the index set of S is D, denote $D_A = \{n \in D\colon S(n) \leq A\}$, $D_B = \{n \in D\colon S(n) \leq B\}$. $\forall n \in D$, $S(n) \leq A \vee B$. Since $S(n) \in M(L^X)$, so $S(n) \leq A$ or $S(n) \leq B$, $D = D_A \cup D_B$. Thus clearly there is at least one of D_A and D_B must be a cofinal subset of D, suppose it is D_A. $\forall n_1, n_2 \in D_A \subset D$, $\exists n_3 \in D$ such that $n_3 \geq n_1, n_2$. Since D_A is a cofinal subset of D, $\exists n_0 \in D_A$ such that $n_0 \geq n_3$. So $n_0 \geq n_1, n_2$, D_A is a directed set. Then we get a subnet $T \in \mathscr{S}_M(\downarrow A)$ of S with the cofinal selection $N\colon D_A \to D$, $N(n) = n$. By (CC2), $T \to_{\mathscr{C}} e$, $e \in \operatorname{clu}_{\mathscr{C}}(A)$.

(CO4). Let $A \in L^X$. By (CO2) proved above, $c(A) \leq c(c(A))$. So we need only prove $\mathrm{clu}_{\mathscr{C}}(c(A)) \subset \mathrm{clu}_{\mathscr{C}}(A)$.

Suppose $\mathscr{A} = \{S \in \mathscr{S}_M(\downarrow A)\colon \ \exists e \in M(L^X), \ (S, e) \in \mathscr{C}\}$. $\forall e \in \mathrm{clu}_{\mathscr{C}}(c(A))$, $\exists S = \{S(n), n \in D\} \in \mathscr{S}_M(\downarrow c(A))$ such that $S \to_{\mathscr{C}} e$. $\forall n \in D$, since $S(n) \in M(\downarrow c(A)) = M(\downarrow \bigvee \mathrm{clu}_{\mathscr{C}}(A))$, by Lemma 4.6, $\exists S^n \in \mathscr{S}_M(\bigcup_{T \in \mathscr{A}} \mathrm{img}(T)) \subset \mathscr{S}_M(\downarrow A)$ such that $S^n \to_{\mathscr{C}} S(n)$. Denote the induced net of $\{S^n\colon n \in D\}$ by $\mathbf{S}$, then $\mathbf{S} \in \mathscr{S}_M(\downarrow A)$ and by (CC4), $\mathbf{S} \to_{\mathscr{C}} e$. By the definition of $\mathrm{clu}_{\mathscr{C}}(A)$, $e \in \mathrm{clu}_{\mathscr{C}}(A)$, $\mathrm{clu}_{\mathscr{C}}(c(A)) \subset \mathrm{clu}_{\mathscr{C}}(A)$. ∎

By the theorem proved above and Theorem 1.12, we can introduce the following.

DEFINITION 4.9. Let X be a nonempty ordinary set, L a F-lattice, $\mathscr{C}$ an L-fuzzy convergence class on L^X. Denote the L-fuzzy topology on X generated by the closure operator on L^X generated by $\mathscr{C}$ as $\psi(\mathscr{C})$.

Then we can establish the following results on the correspondence between L-fuzzy topologies and L-fuzzy convergence classes:

THEOREM 4.10. *Let X be a nonempty ordinary, L a F-lattice. Then*

(i) *For every L-fuzzy topology δ on X, $\psi(\varphi(\delta)) = \delta$.*

(ii) *For every L-fuzzy convergence class $\mathscr{C}$ on L^X, $\varphi(\psi(\mathscr{C})) = \mathscr{C}$.*

(iii) *For every pair δ, μ of L-fuzzy topologies on L^X such that $\delta \subset \mu$, $\varphi(\delta) \supset \varphi(\mu)$.*

(iv) *For every pair $\mathscr{C}, \mathscr{D}$ of L-fuzzy convergence classes on L^X such that $\mathscr{C} \subset \mathscr{D}$, $\psi(\mathscr{C}) \supset \psi(\mathscr{D})$.*

Proof. (i) Suppose the closure operator generated by $\varphi(\delta)$ as c, still denote the closure of $A \in L^X$ in (L^X, δ) as A^-. $\forall U \in \delta$, $(U')^- = U'$. Since $\varphi(\delta)$ is an L-fuzzy convergence class on L^X, by Theorem 4.9, c is a closure operator on L^X. So by (CO1), $U' \leq c(U')$. $\forall e \in M(\downarrow c(U'))$, since $\psi(\delta)$ is an L-fuzzy convergence class, by Lemma 4.6, $\exists S \in \mathscr{S}_M(\downarrow U')$ such that $S \to_{\varphi(\delta)} e$. But this just means $S \to e$ in (L^X, δ), so by Theorem 3.1 (vi) $\Rightarrow$ (i), $e \leq (U')^- = U'$. Therefore, $c(U') \leq U'$, $c(U') = U'$, $U \in \psi(\varphi(\delta))$, $\delta \subset \psi(\varphi(\delta))$.

$\forall U \in \psi(\varphi(\delta))$, $c(U') = U'$. Let S be a molecule net in U', $S \to e \in M(L^X)$. Then $S \to_{\varphi(\delta)} e$, $e \in \mathrm{clu}_{\varphi(\delta)}(U')$, $e \leq \bigvee \mathrm{clu}_{\varphi(\delta)}(U') = c(U') = U'$. By Corollary 3.2 (v) $\Rightarrow$ (i), $(U')^- = U'$, $U \in \delta$. Hence $\psi(\varphi(\delta)) \subset \delta$, $\psi(\varphi(\delta)) = \delta$.

(ii) Denote the closure operator generated by $\mathscr{C}$ as c. $\forall (S, e) \in \mathscr{C}$. If S does not converge to e in $(L^X, \psi(\mathscr{C}))$, then $\exists U \in \mathscr{Q}_{\psi(\mathscr{C})}(e)$ such that S frequently does not quasi-coincide with U, i.e., S is frequently in U'. So S has a subnet $T \in \mathscr{S}_M(\downarrow U')$. In view of (CC2), $T \to_{\mathscr{C}} e$. Hence $e \in \mathrm{clu}_{\mathscr{C}}(U')$,

$e \le c(U') = \mathrm{cl}_{\psi(\mathscr{C})}(U') = U'$, this is a contradiction. So $S \to e$ in $(L^X, \psi(\mathscr{C}))$, $(S, e) \in \varphi(\psi(\mathscr{C}))$. This proves $\mathscr{C} \subset \varphi(\psi(\mathscr{C}))$.

Suppose $(S, e) \in \varphi(\psi(\mathscr{C}))$, then $S \to e$ in $(L^X, \psi(\mathscr{C}))$. We have to prove $S \to_{\mathscr{C}} e$. Suppose this is not true, then by (CC3), S has a subnet $T = \{T(m), m \in E\}$ such that no subnet of T which $\mathscr{C}$-converges to e. By Theorem 2.5(i), $T \to e$ in $(L^X, \psi(\mathscr{C}))$. $\forall m \in E$, denote $E_m = \{k \in E: k \ge m\}$, then E_m is a directed set, $T_m = \{T(k), k \in E_m\}$ is a subnet of T. By $T \to e$ in $(L^X, \psi(\mathscr{C}))$ and Theorem 2.5(i), every $T_m \to e$ in $(L^X, \psi(\mathscr{C}))$. By Theorem 3.1 (vi) $\Rightarrow$ (i),

$$e \le \mathrm{cl}_{\psi(\mathscr{C})}\big(\bigvee \mathrm{img}(T_m)\big) = c\big(\bigvee \mathrm{img}(T_m)\big) = \bigvee \mathrm{clu}_{\mathscr{C}}\big(\bigvee \mathrm{img}(T_m)\big).$$

By Lemma 4.6, $\exists R_m \in \mathscr{S}_M(\downarrow \bigvee \mathrm{img} T_m))$ such that $R_m \to_{\mathscr{C}} e$. By Lemma 4.7, $\exists \mathbf{R}_m \in \mathscr{S}_M(\mathrm{img}(T_m))$ and $\mathbf{R}_m \to_{\mathscr{C}} e$. Take $K(m) = e$ for every $m \in E$, then by (CC1), $K = \{K(m), m \in E\} \to_{\mathscr{C}} e$, $\forall m \in E$, $\mathbf{R}_m \to_{\mathscr{C}} K(m)$. By (CC4), the induced net $\mathbf{K}$ of $\{\mathbf{R}_m: m \in E\}$ $\mathscr{C}$-converges to e.

For the index set $E \times \Pi_{m \in E} D_m$ of $\mathbf{K}$ and an arbitrary element $(m, f) \in E \times \Pi_{m \in E} D_m$, since $\mathbf{K}(m, f) = \mathbf{R}_m(f(m)) \in \mathrm{img}(T_m)$, $\exists N(m, f) \in E$ such that $N(m, f) \ge m$ and $\mathbf{K}(m, f) = T(N(m, f))$. So we obtain a mapping $N: E \times \Pi_{m \in E} D_m \to E$ such that $\mathbf{K} = T \circ N$. $\forall m_0 \in E$, arbitrarily fix $f_0 \in \Pi_{m \in E} D_m$, then $\forall (m, f) \in E \times \Pi_{m \in E} D_m$ such that $(m, f) \ge (m_0, f_0)$, $N(m, f) \ge m \ge m_0$. Therefore, N is a confinal selection on T, $\mathbf{K}$ is a subnet of T. But we have proved above that $\mathbf{K} \to_{\mathscr{C}} e$, this contradicts with the property of T that it has no subnet which $\mathscr{C}$-converges to e.

(iii) Suppose δ, μ are two L-fuzzy topologies on L^X and $\delta \subset \mu$. If $(S, e) \in \varphi(\mu)$, then $S \to e$ in (L^X, μ), i.e., S eventually quasi-coincides with every element of $\mathscr{Q}_\mu(e)$. But $\delta \subset \mu$, so $\mathscr{Q}_\mu(e) \supset \mathscr{Q}_\delta(e)$. Hence S eventually quasi-coincides with every element of $\mathscr{Q}_\delta(e)$, $S \to e$ in (L^X, δ), $(S, e) \in \varphi(\delta)$, $\varphi(\mu) \subset \varphi(\delta)$.

(iv) Suppose $\mathscr{C}, \mathscr{D}$ are L-fuzzy convergence classes on L^X and $\mathscr{C} \subset \mathscr{D}$, then for every $A \in L^X$, we have

$$\begin{aligned}
\mathrm{clu}_{\mathscr{C}}(A) &= \{e \in M(L^X): \exists S \in \mathscr{S}_M(\downarrow A), S \to_{\mathscr{C}} e\} \\
&= \{e \in M(L^X): \exists S \in \mathscr{S}_M(\downarrow A), (S, e) \in \mathscr{C}\} \\
&\subset \{e \in M(L^X): \exists S \in \mathscr{S}_M(\downarrow A), (S, e) \in \mathscr{D}\} \\
&= \{e \in M(L^X): \exists S \in \mathscr{S}_M(\downarrow A), S \to_{\mathscr{D}} e\} \\
&= \mathrm{clu}_{\mathscr{D}}(A).
\end{aligned}$$

So let $\mathrm{cl}_{\mathscr{C}}$ and $\mathrm{cl}_{\mathscr{D}}$ denote the closure operators generated by $\mathscr{C}$ and $\mathscr{D}$ respectively, for an arbitrary closed subset A in L-fts $(L^X, \psi(\mathscr{D}))$, we have

$$\mathrm{cl}_{\mathscr{C}}(A) = \bigvee \mathrm{clu}_{\mathscr{C}}(A) \le \bigvee \mathrm{clu}_{\mathscr{D}}(A) = \mathrm{cl}_{\mathscr{D}}(A) = A,$$

i.e., $\mathrm{cl}_{\mathscr{C}}(A) = A$, A is also closed in $(L^X, \psi(\mathscr{C}))$. This exactly means $\psi(\mathscr{C}) \supset \psi(\mathscr{D})$. ∎

Remark 4.11. It is not difficult to show that (CC5) can be equivalently replaced by

(v′) For every $\mathscr{A} \subset M(L^X)$ and every $e \in M(\downarrow \bigvee \mathscr{A})$, there exists $S \in \mathscr{S}_M(\mathscr{A})$ such that $S \to_{\mathscr{C}} e$. (CC5′)

It is just the essential difference between convergence classes in L-fuzzy topological spaces and general topological spaces. For the reason, (CC5) can not be omitted from the definition of L-fuzzy convergence class (Example 4.13), although it will become a trivial condition in general topology.

EXAMPLE 4.12. Take X be a singleton $\{x\}$, $L = [0,1]$. For every net $S = \{S(n), n \in D\}$ on L^X, denote $F(S) = \{ht(S(n)), n \in D\}$, then $F(S)$ is a net in $[0,1]$. Use S to denote a net in L^X, take

$$\mathscr{C} = \left\{(S, x_\lambda)\colon \lambda \in [0,1]\backslash\left\{\tfrac{2}{3}, 1\right\} F(S) \text{ eventually equals to } \lambda\right\}$$
$$\cup = \left\{\left(S, x_{\frac{2}{3}}\right)\colon \forall \varepsilon > 0,\ F(S) \text{ eventually is in } \left(\tfrac{1}{3} - \varepsilon, \tfrac{1}{3}\right) \cup \left\{\tfrac{2}{3}\right\}\right\}$$
$$\cup = \left\{(S, x_1)\colon \forall \varepsilon > 0,\ F(S) \text{ eventually is in } \left(\tfrac{2}{3} - \varepsilon, \tfrac{2}{3}\right) \cup \{1\}\right\},$$

then $\mathscr{C} \subset \mathscr{S}_M(L^X) \times M(L^X)$ and clearly satisfies (CC1)–(CC3).

Suppose $S = \{S(n), n \in D\}$ is a net in L^X, $S \to_{\mathscr{C}} x_\lambda$, $\forall n \in D$, T^n is a net in L^X, $T^n \to_{\mathscr{C}} S(n)$.

If $\lambda \in [0,1]\backslash\{\frac{2}{3}, 1\}$, then $F(S)$ eventually equals to $\lambda \in [1,1]\backslash\{\frac{2}{3}, 1\}$, so for every $n \in D$, $F(T^n)$ eventually equals to λ. Hence the induced net of $\{T^n\colon n \in D\}$ $\mathscr{C}$-converges to x_λ.

If $\lambda = \frac{2}{3}$, then for every $\varepsilon > 0$, $F(S)$ eventually is in $(\frac{1}{3} - \varepsilon, \frac{1}{3}) \cup \{\frac{2}{3}\}$. $\forall n \in D$,

(1) If $F(S)(n) \in (\frac{1}{3} - \varepsilon, \frac{1}{3})$, then $F(T^n)$ eventually equals to $F(S)(n)$;

(2) If $F(S)(n) = \frac{2}{3}$, then $F(T^n)$ eventually is in $(\frac{1}{3} - \varepsilon, \frac{1}{3}) \cup \{\frac{2}{3}\}$.

So for every $n \in D$, $F(T^n)$ always eventually is in $(\frac{1}{3} - \varepsilon, \frac{1}{3}) \cup \{\frac{2}{3}\}$, this implies that the induced net of $\{T^n, n \in D\}$ $\mathscr{C}$-converges to $x_{2/3} = x_\lambda$.

If $\lambda = 1$, the discussion is similar.

So we have proved that the induced net of $\{T^n, n \in D\}$ always $\mathscr{C}$-converges to (CC4) is true for $\mathscr{C}$.

Now for the closure operator c generated by $\mathscr{C}$, clearly $c(x_{1/3} = x_{2/3}$, $cc(x_{1/3}) = c(x_{2/3}) = x_1$, the condition (CO4) is not satisfied, c is not a closure operator on L^X.

REFERENCES

1. G. Gerla, On fuzzy convergence, *Info. Sci.* **39**, No. 3 (1986), 269–284.
2. Cheng-Ming Hu, Theory of convergence in fuzzy topological spaces, *J. Fuzzy Math.* **1**, No. 1 (1993), 1–12.
3. B. Hutton, Uniformities on fuzzy topological spaces, *J. Math. Anal. Appl.* **58** (1977), 559–571.
4. E. Lowen and R. Lowen, Characterization of convergence in fuzzy topological spaces, *Internat. J. Math. Math. Sci.* **8**, No. 3 (1985), 497–511.
5. E. Lowen, R. Lowen, and P. Wuyts, The categorical topology approach to fuzzy topology and fuzzy convergence, *Fuzzy Sets and Systems* **40**, No. 2 (1991), 347–373.
6. Ying-Ming Liu, Intersection operation on union-preserving mappings in completely distributive lattices, *J. Math. Anal. Appl.* **84** (1981), 249–255.
7. Ying-Ming Liu, On fuzzy convergence classes, *Fuzzy Sets and Systems* **30**, (1989), 47–51.
8. Ying-Ming Liu, Fuzzy topology, stratifications and category theory, *in* "Between Mind and Computer," (Wang Loe, Ed.), 1993, Singapore.
9. Bao-ming Pu and Ying-Ming Liu, Fuzzy topology I, neighborhood structure of a fuzzy point, *J. Sichuan Univ.*, Natural Sci. Edition, 1977, No. 1, 31–50 (in Chinese), and English edition see *J. Math. Anal. Appl.* **76** (1980), 541–599.
10. Guo-jun Wang, Pointwise topology on completely distributive lattices, *Fuzzy Sets and Systems*, **30**, No. 1 (1980), 53–62.
11. Guo-jun Wang, On the structure of fuzzy lattices, *Acta Math. Sinica*, **4** (1986), 539–543.
12. Guo-jun Wang, *Theory of L-fuzzy Topological Spaces*, Shaanxi Normal University Press, 1988, (in Chinese).
13. Dong-Sheng Zhao, The N-compactness in L-fuzzy topological spaces, *J. Math. Anal. Appl.* **128** (1987), 64–79.